Multibunch solutions of the differential-difference equation for traffic flow

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The Newell-Whitham type of car-following model, with a hyperbolic tangent as the optimal velocity function, has a finite number of exact steady traveling wave solutions that can be expressed in terms of elliptic theta functions. Each such solution describes a density wave with a definite number of car bunches on a circuit. In our numerical simulations, we observe a transition process from uniform flow to congested flow described by a one-bunch analytic solution, which appears to be an attractor of the system. In this process, the system exhibits a series of transitions through which it comes to assume configurations closely approximating multi-bunch solutions with successively fewer bunches.

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I. INTRODUCTION

Traffic flow on no-passing freeways has been extensively studied using car-following models. Some of these models can describe the spontaneous generation of density waves, i.e., traffic congestion, in the collective motion of cars. The optimal velocity (OV) model [1,2] reveals many features of congested flow. The equation of motion for this model is

$$\ddot{x}_n(t)=a[V(\Delta x_n(t))-\dot{x}_n(t)]. \tag{1}$$

Here, x_n and Δx_n denote the position and the ''headway'' $(x_{n-1}-x_n)$ of the nth car, respectively. Each driver of a car attempts to adjust its velocity to the optimal velocity $V(\Delta x)$, as determined by its headway. The constant a parametrizes the sensitivity of the adjustment, which must be less than some critical value for congestion to result. It has been found that in a certain range of values of the mean headway, cars gradually form bunches, and the positions of these bunches move backward with a constant velocity.

Most studies have been made using numerical simulations with cyclic boundary conditions. In these studies it is observed that high density regions, the bunches, appear in alternation with low density regions, forming a density wave of congested flow. Within a given region, all cars move with almost constant headway and velocity, while in the interfaces between regions the density changes rapidly, exhibiting a characteristic kink shape. The shape of the kink at an interface seems to be determined only by the sensitivity a and the OV function $V(\Delta x)$, irrespective of the initial conditions and the mean headway.

There is very little known about analytical solutions of this model. In the vicinity of the critical value of a, the interface can be approximately described by the kink solution of the modified Korteweg–de Vries equation [3]. For some particular choices of the OV function, exact solutions that describe the interface have been obtained [4,5]. However, in numerical simulations, the evolution process of the congestion is quite complicated. For example, a number of bunches with various lengths may arise in the circuit, and these bunches often fuse. With the present understanding, it is not possible to predict the number or lengths of the bunches that will appear, nor is it possible to describe the processes through which they fuse.

In contrast to Eq. (1), the traditional car-following model [6], which consists of the first-order differential difference equation

$$\dot{x}_n(t+\tau)=V(\Delta x_n(t)), \tag{2}$$

has been studied in the field of traffic engineering for several decades. The reaction time τ here is a time lag which represents the time it takes a car to respond to a change in motion. The models (1) and (2) are quite similar in their qualitative behavior, especially with regard to the generation of density waves. Equation (2) can actually be reduced to Eq. (1) by truncating the Taylor expansion of its left-hand side as $\dot{x}_n(t+\tau)\simeq\dot{x}_n(t)+\tau\ddot{x}_n(t)$. Numerical simulations show that Eq. (2), with $V(\Delta x)$ given by a hyperbolic tangent, possesses congested flow solutions quite similar to those for the corresponding OV model [7].

Early in the 1990s, Whitham [8] found that the model (2) with an exponential OV function has a pulselike, exact steady traveling wave solution that can be written in terms of elliptic functions. Although with his choice of the OV function there are no car bunching solutions, he pointed out a condition on the relation between the time lag τ and the propagation velocity necessary for the existence of such solutions [see Eq. (5) below]. We call this the Whitham condition. Recently [9], it was demonstrated that the model (2) with the hyperbolic tangent OV function

$$V(\Delta x)=\xi+\eta\tanh\left(\frac{\Delta x-\rho}{2\sigma}\right) \tag{3}$$

has a class of car bunching solutions that can be written in terms of elliptic theta functions. The existence of such solutions requires the Whitham condition to be satisfied. The authors of Ref. [10] report that the Whitham condition is satisfied in numerical simulations for a much more general class of OV functions.

In this paper, we investigate the structure of the class of exact traveling wave solutions of Eq. (2). With cyclic boundary conditions, each solution represents a periodic density wave with definite length and number of bunches. Generally, there exist several solutions with different numbers of bunches. We refer to these as the ''multibunch solutions.'' We also report the results of numerical simulations in which we observed the generation process of density waves. We found that this process occurs through relaxation to one of the analytic solutions.

In the next section, we investigate the parameter space of the exact solutions and find a finite series of multibunch solutions. In Sec. III, the parameter values that allow multibunch solutions are determined. These values were used in the numerical simulations reported in Sec. IV. The final section is devoted to a summary and discussion. Some mathematical formulas for the elliptic functions and the exact solutions are summarized in the Appendixes.

II. MULTIBUNCH SOLUTIONS IN A CIRCUIT

We begin by summarizing our previous results [9] for the exact solution with width parameter δ,

$$x_n(t)=Ct-nh+A\ln\frac{\vartheta_0(\nu t-2\beta n-\beta+\delta,q)}{\vartheta_0(\nu t-2\beta n-\beta-\delta,q)},\qquad(4)$$

where $\vartheta_0(v,q)$ is the elliptic theta function, and q, A, β, ν, δ, C, and h are free parameters that characterize the solution. Here, q is the modulus parameter of the theta function, and h is the mean headway of the N cars in the circuit, whose length is $L\equiv Nh$. The traffic flow expressed by Eq. (4) displays the alternate appearance of *high density regions* and *low density regions*. In this way, the traffic in the circuit is divided into several bunches, which move backward. The cyclic boundary condition $x_{n+N}(t)=x_n(t)-Nh$, together with the periodicity of the theta function, $\vartheta_0(v+1,q)=\vartheta_0(v,q)$, implies that $2\beta N$ must be an integer, which coincides with the number of bunches n_b. We call β the ''bunch parameter.'' The wavelength $1/(2\beta)$ is approximately equal to the number of cars within a pair of consecutive high and low density regions. The width parameter δ, which ranges in $0<2\delta<1$, determines the proportion of low density region in a wavelength: $2\delta/(2\beta)$ and $(1-2\delta)/(2\beta)$ are roughly equal to the number of cars in the low and high density regions, respectively.

The expression (4), which satisfies the equation of motion (2) with the OV function (3), gives us a finite set of exact solutions. As discussed in Ref. [9], the traffic model with a given time lag constant τ possesses an exact solution of the form (4) only if the Whitham condition

$$\nu\tau=\beta\qquad(5)$$

is met. Under this condition, there are four relations between the free parameters and the coefficients of the equation of motion ξ, η, ρ, and σ, as described in Appendix B [see Eqs. (B14)–(B17)]. First, the mean headway h is determined by the system size L and N. Then, ν is fixed in terms of τ, and β by the Whitham condition. We also find that A is identical with σ, as given in Eq. (B17). Finally, the remaining four parameters q, β, δ, and C are subject to the three relations (B14)–(B16). It would appear that a parameter remains undetermined and that the system has a one-parameter family of exact solutions. However, noting that β is discretized, we see that the bunch number n_b must be an integer. Moreover, the number of bunches cannot exceed the total number of cars N. (We see below that a much stronger restriction in fact exists for the maximum number of bunches.) Consequently, the form (4) represents a *finite* number of exact solutions, which describe the n_b-bunch configurations of traffic congestion. These are the multibunch solutions mentioned above.

To examine the multibunch solutions, let us analyze the three relations (B14)–(B16) in detail:

$$\xi=C+\frac{\sigma\beta}{2\tau}\frac{d}{d\beta}\ln\frac{\vartheta_1(2\delta+\beta)}{\vartheta_1(2\delta-\beta)},\qquad(6)$$

$$\eta=\frac{\sigma\beta}{2\tau}\frac{d}{d\beta}\ln\frac{\vartheta_1^2(\beta)}{\vartheta_1(2\delta+\beta)\vartheta_1(2\delta-\beta)},\qquad(7)$$

$$\rho=h-\sigma\ln\frac{\vartheta_1(2\delta-\beta)}{\vartheta_1(2\delta+\beta)}.\qquad(8)$$

When the time lag τ and the coefficients ξ, η, ρ, and σ are given, we can solve these three equations. Equation (7) gives a relation among q, β, and δ. By solving this equation, we can express δ as a function of q and β. In this way, we obtain a solvability condition for δ, which gives us an allowed region for the parameter q for each possible value of β. The parameters C and h can also be computed as functions of q and β by substituting the expression for $\delta(q,\beta)$ into Eqs. (6) and (8). For each possible β, the modulus parameter q is determined so as to satisfy the equation $h(q,\beta)=L/N$.

First, we rewrite Eq. (7) in terms of the Jacobi elliptic function. The product of the theta functions in Eq. (7) can be decomposed by using the addition formula given in Appendix A. Replacing $\vartheta_1(v)/\vartheta_0(v)$ by the Jacobi elliptic function $\sqrt{k}\,\mathrm{sn}\,2Kv$ in Eq. (B19), we rewrite it as

$$\frac{\tau}{\tau_c}=-\frac{\beta}{2}\frac{d}{d\beta}\ln\left(\frac{1}{\mathrm{sn}^2\,2K\beta}-\frac{1}{\mathrm{sn}^2\,4K\delta}\right).\qquad(9)$$

Here σ/η is denoted by τ_c for the following reason. A linear analysis [8,11] gives the instability condition

$$2\tau V'(h)\equiv\tau\frac{\eta}{\sigma}\mathrm{sech}^2\left(\frac{h-\rho}{2\sigma}\right)>\frac{\pi/N}{\sin\pi/N}\qquad(10)$$

for uniform flow described by $x_n^{(0)}(t)=V(h)t-nh$. [Note that the right-hand side of Eq. (10) is almost equal to 1, provided that N is not too small.] Then the time lag τ must satisfy the condition

$$\tau>\tau_c\equiv\frac{\sigma}{\eta},\qquad(11)$$

in order for there to exist uniform flow that decays in such a way as to produce congested flow. The value τ_c thus represents the critical time lag.

We now discuss the solvability condition for Eq. (9) and the allowed region for q and β. By differentiating with respect to β, we can solve Eq. (9) for $\mathrm{sn}\,4K\delta$. We obtain

$$\mathrm{sn}^2\,4K\delta=\frac{\mathrm{sn}^2\,2K\beta}{1-(\tau_c/\tau)(2K\beta\,\mathrm{cn}\,2K\beta\,\mathrm{dn}\,2K\beta/\mathrm{sn}\,2K\beta)}.\qquad(12)$$

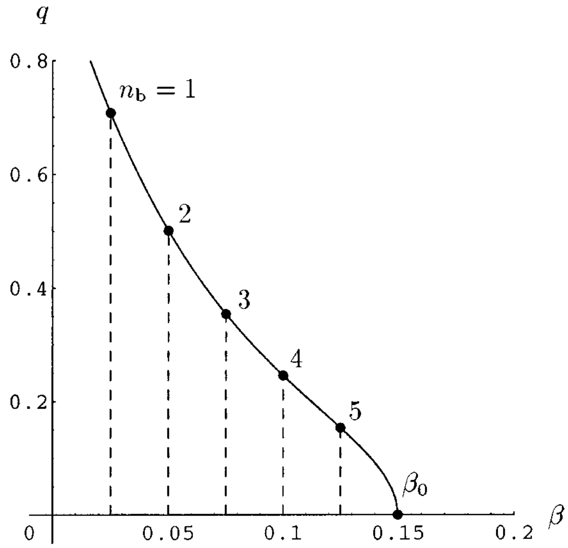

FIG. 1. The allowed region as determined by the solvability condition with $\tau_c/\tau=0.858\,69$. The maximum value of the bunch parameter β_0 is 0.149 835. The vertical dashed lines indicate the ranges of q for some possible values of β with $N=20$. These correspond to $n_b=1$, 2, 3, 4, and 5.

Since $\mathrm{sn}^2\,4K\delta\leq 1$, the right-hand side of the above equation cannot exceed 1. Thus, the solvability condition that guarantees the existence of δ is given by

$$\frac{\tau_c}{\tau}\leq\frac{\mathrm{sn}\,2K\beta\,\mathrm{cn}\,2K\beta}{2K\beta\,\mathrm{dn}\,2K\beta}. \tag{13}$$

This inequality gives us a restriction on the number of bunches for a given τ. Since both the complete elliptic integral of the first kind, K, and the modulus of the Jacobi elliptic functions, k, are functions of q [as given in Eq. (A3)], the right-hand side of Eq. (13) depends only on β and q. In Fig. 1, we show the allowed region determined by Eq. (13) in the (β,q) plane for the case $\tau_c/\tau=0.858\,69$. It can easily be checked that the allowed region actually disappears for $\tau<\tau_c$. The boundary curve in Fig. 1 crosses the β axis at β_0, the maximum value of β, which satisfies[1]

$$\frac{\tau_c}{\tau}=\frac{\sin 2\pi\beta_0}{2\pi\beta_0}. \tag{14}$$

Thus, β_0 is determined by only the value of τ_c/τ. The existence of a nontrivial upper bound of β implies that the number of bunches is restricted much more severely than by the obvious condition $n_b<N$. Explicitly, the maximum value of the number of bunches $n_b^{\max}$, obtained from Eq. (14), is given by

$$n_b^{\max}=[2N\beta_0], \tag{15}$$

[1]The right-hand side of Eq. (14) is given by the $q\to 0$ limit of the right-hand side of Eq. (13).

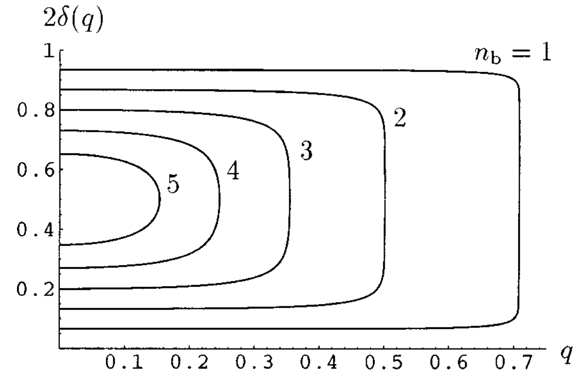

FIG. 2. The width parameter $2\delta_\pm(q)$ for $\tau_c/\tau=0.858\,69$ with $N=20$. Each curve corresponds to a possible bunch number n_b appearing in Fig. 1.

where $[x]$ is the maximal integer that does not exceed x. Consequently, when the time lag τ and the total car number N are given, the system has $n_b^{\max}$ multibunch solutions with the bunch parameters

$$\beta=\frac{1}{2N},\frac{2}{2N},\ldots,\frac{n_b}{2N},\ldots,\frac{n_b^{\max}}{2N}. \tag{16}$$

When we specify one of the possible bunch numbers, or equivalently one of the allowed values of β, the solvability condition (13) gives us an allowed range of the modulus parameter q:

$$0\leq q\leq q_{\max}(\beta). \tag{17}$$

Here $q_{\max}(\beta)$ is the value corresponding to equality in the solvability condition for a given value of β. The vertical dashed lines in Fig. 1 indicate the ranges of q for several values of β with $N=20$. In the case depicted there, $n_b^{\max}=5$.

III. CONSTRUCTION OF A MULTIBUNCH SOLUTION

In this section, we determine the width parameter δ and the modulus parameter q for a fixed n_b (or β) to construct the n_b-bunch solution. In the allowed range (17), Eq. (12) has two branches of δ as functions of q. One, which we denote δ_-, remains in the range $0<2\delta_-<\frac{1}{2}$ and is given by

$$2\delta_-(q)=\frac{1}{2K}\mathrm{sn}^{-1}\times\frac{\mathrm{sn}\;2K\beta}{\sqrt{1-(\tau_c/\tau)(2K\beta\;\mathrm{cn}\;2K\beta\;\mathrm{dn}\;2K\beta/\mathrm{sn}\;2K\beta)}}, \tag{18}$$

where the branch of the inverse Jacobi function sn^{-1} is selected as $0<\mathrm{sn}^{-1}<K$. The other branch, which we denote δ_+, remains in the range $\frac{1}{2}<2\delta_+<1$ (corresponding to $K<\mathrm{sn}^{-1}<2K$) and is given by $2\delta_+(q)\equiv 1-2\delta_-(q)$. These two branches $2\delta_-(q)$ and $2\delta_+(q)$ are connected at $q=q_{\max}(\beta)$, where $2\delta=\frac{1}{2}$, as depicted in Fig. 2.

Corresponding to these branches, the mean headway $h(q)$ also has two branches $h_-<\rho$ and $h_+>\rho$. By putting $2\delta_\pm(q)$ into Eq. (8), we obtain an expression for each branch:

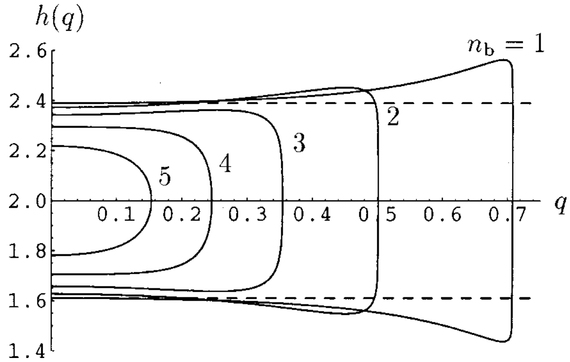

FIG. 3. The mean headway $h_{\pm}(q)$ for $\tau_c/\tau=0.858\,69$, $\rho=2$, and $\sigma=1/2$ with $N=20$. Each curve corresponds to a possible bunch number n_b appearing in Figs. 1 and 2. The horizontal dashed lines indicate the critical values of the headway as determined by the linear instability, which coincide with the $q\to0$ limits of $h(q)$ for $n_b=1$.

$$h_{\pm}(q)=\rho+\sigma\ln\frac{\vartheta_1(2\delta_{\pm}(q)-\beta,q)}{\vartheta_1(2\delta_{\pm}(q)+\beta,q)}. \tag{19}$$

Since $h_-(q)+h_+(q)=2\rho$, these functions are symmetric with respect to reflection about $h=\rho$. The combined entire function $h(q)$ is displayed in Fig. 3. We can now determine the modulus parameter q, within a certain range of values of the mean headway L/N, by solving $h(q)=L/N$. Once q is obtained, the parameter C is calculated directly from Eq. (6). In this way, the exact multibunch solution (4) can be constructed. For the first branch, $h_-<\rho$, the width parameter is in the range $0<2\delta<\frac{1}{2}$, and the low density region in the traffic flow contains more cars than the high density region. Contrastingly, for the second branch $h_+>\rho$, the width parameter exceeds $\frac{1}{2}$, and the congested region has more cars. A ''symmetric density wave,'' corresponding to $2\delta=\frac{1}{2}$, always exists when the mean headway h is equal to ρ, the inflection point of the OV function, since $\vartheta_1(\frac{1}{2}+\beta)=\vartheta_1(\frac{1}{2}-\beta)$, regardless of the values of β and q.

When the equation $h(q)=L/N$ allows multiple values of q, although an analytic solution can be constructed for each q, not all of these solutions are stable. In Fig. 3, the dashed lines indicate the linearly unstable region [determined by Eq. (10)] of the (common) headway of the uniform flow $x_n^{(0)}(t)$:

$$|h-\rho|<2\sigma\,\mathrm{arccosh}\sqrt{\frac{\tau}{\tau_c}\,\frac{\sin\pi/N}{\pi/N}}. \tag{20}$$

Outside this range, we find two different values of q for $n_b=1$ or 2. In this case, the solution corresponding to uniform flow is linearly stable. However, we cannot say for certain whether multibunch solutions are stable. We will discuss this point in another work [12]. On the other hand, it is likely that unstable uniform flow will develop into one of the analytic solutions within the range (20). However, the process in which a multibunch solution is generated from a given initial configuration is very complicated. For this reason, it is difficult to predict which of the possible analytic solutions will be selected.

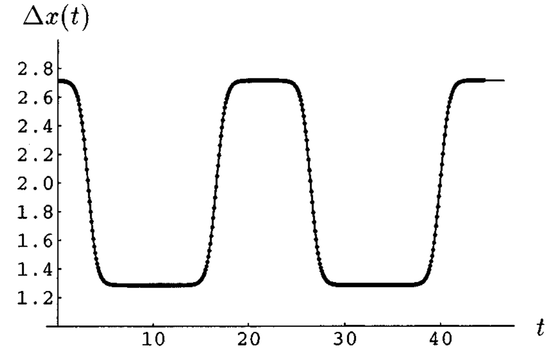

FIG. 4. The result of a numerical simulation with $N=20$, $\tau_c/\tau=0.858\,69$, $V(\Delta x)=\tanh(\Delta x-2)+\tanh 2$ after a sufficient relaxation time $t\simeq6\times10^4$. The curve plotted by the dots represents the results of the numerical simulation, while the solid line represents the analytic one-bunch solution with $q=0.707\,921\,403\,287\,55$.

IV. NUMERICAL SIMULATIONS

To investigate the process in which a multibunch solution is generated from uniform flow, we performed numerical simulations for the differential-difference equation (2). We used the OV function

$$V(\Delta x)=\tanh(\Delta x-2)+\tanh 2. \tag{21}$$

That is, the coefficients were chosen as $\xi=\tanh 2$, $\eta=1$, $\rho=2$, and $\sigma=0.5$ ($\tau_c=0.5$). The time lag $\tau=0.588\,28$ gives $\tau_c/\tau=0.858\,69<1$. We prepared 20 cars and arranged them so that they formed a uniform flow with constant headway $h=1.885\,71$ and initial velocity $V(h)=0.850\,233$ in a circuit whose circumference was $L=37.7142$. The uniform flow was maintained for a duration τ to prepare the initial function of the differential-difference equation. The solvability condition (13) tells us that the maximum number of bunches in this case is five. The values of the modulus parameter q for the possible bunch numbers $n_b=1$, 2, 3, 4, and 5 are 0.707 921 403 287 55, 0.501 133 76, 0.353 616 7, 0.241 804 4, and 0.140 292, respectively. Since uniform flow is linearly unstable, a density wave begins to develop. In this case, the one-bunch mode was ultimately generated after a sufficient relaxation time $t\simeq6\times10^4$. The result, which is represented in Fig. 4 by dots, agrees quite well with the analytic one-bunch solution with $q=0.707\,921\,403\,287\,55$, as represented by the thin line in the figure.

We observed many interesting phenomena in the process in which the one-bunch solution eventually appears. First, until $t\simeq300$, the initial uniform flow developed into a three-bunch configuration which closely resembles the exact three-bunch solution. Although this configuration maintained its shape for some time, it gradually became distorted, with one of the bunches moving closer to a neighboring bunch. At $t\simeq4680$, the fusion of these bunches occurred, and system transformed into a two-bunch configuration. The two-bunch configuration existed about ten times longer than the three-bunch one. However, eventually, one of the bunches began to shrink. This bunch was finally absorbed by the other at $t\simeq51\,560$, as shown in Fig. 5, and the formation of the one-bunch solution was completed. The simulation was continued until $t=2\times10^5$, with the one-bunch solution continuing

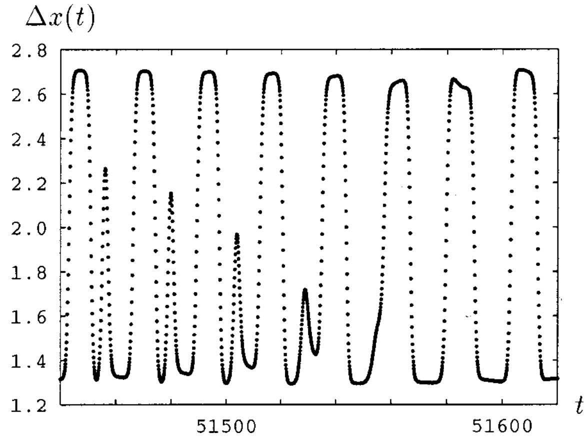

FIG. 5. The fusion process at $t \cong 51\,560$. One bunch begins to shrink at $t \cong 48\,000$ and is absorbed by the other.

to survive, unchanged in form. These results suggest that the one-bunch solution is an attractor of the system [13].

V. SUMMARY AND DISCUSSION

We found a finite set of exact multibunch solutions for a car-following model consisting of a differential-difference equation with a cyclic boundary condition characterized by a driver's reaction time. We found that when the circuit length and the total number of cars are given, we can calculate the possible number of car bunches in the circuit and the profile of the density wave that describes the car bunching. Our numerical simulations show that linearly unstable uniform flow gradually develops into a one-bunch solution, which seems to be an attractor of the system. In the relaxation process from uniform flow to the one-bunch solution, the system remains for extended times in configurations that closely approximate higher multibunch solutions. We found that the durations of these quasistable configurations increase exponentially as the number of bunches decreases (see Fig. 1 in Ref. [13]).

The series of transitions from one multibunch configuration to the next suggests that each multibunch solution corresponds to a heteroclinic point of the system. However, no flow out of the one-bunch solution was observed, which suggests that it is an attractor. It is also possible that each of the multibunch solutions is a kind of Milnor attractor [14], which is unstable with respect to any small perturbation, but globally attracts orbits. To fully understand the situation we must investigate the stability and attracting domain of the multibunch solutions more precisely.

The formation of density waves from uniform flow in the OV model is very similar to that in the present model. Car bunching and the fusion of bunches have also been observed in the OV model. It is reasonable to believe that the existence of multibunch solutions and the qualitative features of the cascade of transitions are common to both models, although at this time we cannot be sure, as analytic solutions for the OV model have not yet been obtained.

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APPENDIX A: DEFINITIONS AND FORMULAS OF ELLIPTIC FUNCTIONS

The definitions of elliptic functions and some mathematical formulas used in this paper are listed in this Appendix.

The elliptic theta functions are defined as the infinite products

$$\vartheta_0(v,q) = q_0 \prod_{n=1}^{\infty} (1 - 2q^{2n-1}\cos 2\pi v + q^{4n-2}), \tag{A1}$$

$$\vartheta_1(v,q) = 2\pi^{1/4} q_0 \sin \pi v \prod_{n=1}^{\infty} (1 - 2q^{2n}\cos 2\pi v + q^{4n}),$$

where $q_0 = \Pi_{n=1}^{\infty}(1-q^{2n})$. The addition formulas for the theta functions are

$$\vartheta_0(v+w)\,\vartheta_0(v-w)\,\vartheta_0^2(0) = \vartheta_0^2(v)\,\vartheta_0^2(w) - \vartheta_1^2(v)v_1^2(w), \tag{A2}$$

$$\vartheta_1(v+w)\,\vartheta_1(v-w)\,\vartheta_0^2(0) = \vartheta_1^2(v)\,\vartheta_0^2(w) - \vartheta_0^2(v)\,\vartheta_1^2(w).$$

The Jacobi elliptic functions, the modulus k, and the complementary modulus k' can be expressed in terms of the theta functions as follows:

$$k = \frac{\vartheta_2^2(0,q)}{\vartheta_3^2(0,q)}, \quad k' = \frac{\vartheta_0^2(0,q)}{\vartheta_3^2(0,q)}, \quad K = \frac{\pi}{2}\vartheta_3^2(0,q),$$

$$\mathrm{sn}\ 2Kv = \frac{1}{\sqrt{k}}\frac{\vartheta_1(v)}{\vartheta_0(v)}, \quad \mathrm{cn}\ 2Kv = \sqrt{1-\mathrm{sn}^2 2Kv}, \tag{A3}$$

$$\mathrm{dn}\ 2Kv = \sqrt{1-k^2\mathrm{sn}^2\ 2Kv},$$

where K is the complete elliptic integral of the first kind.

APPENDIX B: THETA FUNCTION FORMALISM

Let us solve the equation of motion

$$\dot{x}_n(t+\tau) = \xi + \eta \tanh\left(\frac{\Delta x_n(t) - \rho}{2\sigma}\right). \tag{B1}$$

with the assumed form

$$x_n(t) = Ct - nh + A \ln \frac{\vartheta_0(v-\beta+\delta)}{\vartheta_0(v-\beta-\delta)}, \tag{B2}$$

where

$$v \equiv \nu t - 2\beta n. \tag{B3}$$

By virtue of the Whitham condition $\nu\tau = \beta$, the velocity at $t+\tau$ can be written as

$$\dot{x}_n(t+\tau) = C + A\nu \frac{d}{dv} \ln \frac{\vartheta_0(v+\delta)}{\vartheta_0(v-\delta)}, \tag{B4}$$

where we have replaced the time differentiation d/dt with $\nu\, d/dv$. Then, converting the v differentiation into $d/d\delta$, we can apply the addition formula (A2) to the expression

$$\frac{d}{dv}\ln\frac{\vartheta_0(v+\delta)}{\vartheta_0(v-\delta)}=\frac{d}{d\delta}\ln[\vartheta_0(v+\delta)\vartheta_0(v-\delta)]$$
$$=\frac{d}{d\delta}\ln[\vartheta_0^2(v)\vartheta_0^2(\delta)-\vartheta_1^2(v)\vartheta_1^2(\delta)]. \quad \text{(B5)}$$

This allows us to write the velocity in a rational expression of $\vartheta_0^2(v)$ and $\vartheta_1^2(v)$:

$$\dot{x}_n(t+\tau)=C+A\nu\frac{\vartheta_0^2(v)[\vartheta_0^2(\delta)]'-\vartheta_1^2(v)[\vartheta_1^2(\delta)]'}{\vartheta_0^2(v)\vartheta_0^2(\delta)-\vartheta_1^2(v)\vartheta_1^2(\delta)}. \quad \text{(B6)}$$

We find that the headway

$$\Delta x_n(t)=h+A\ln\frac{\vartheta_0(v+\beta+\delta)\vartheta_0(v-\beta-\delta)}{\vartheta_0(v+\beta-\delta)\vartheta_0(v-\beta+\delta)} \quad \text{(B7)}$$

can also be expressed in a rational form with $\vartheta_0^2(v)$ and $\vartheta_1^2(v)$:

$$e^{2X_n}=\frac{\vartheta_0^2(v)\vartheta_0^2(\delta+\beta)-\vartheta_1^2(v)\vartheta_1^2(\delta+\beta)}{\vartheta_0^2(v)\vartheta_0^2(\delta-\beta)-\vartheta_1^2(v)\vartheta_1^2(\delta-\beta)}. \quad \text{(B8)}$$

Here

$$X_n=\frac{\Delta x_n(t)-h}{2A}. \quad \text{(B9)}$$

By eliminating $\vartheta_0^2(v)$ and $\vartheta_1^2(v)$ from Eqs. (B6) and (B8), it can be shown that the velocity $\dot{x}_n(t+\tau)$ is equal to a first order rational expression of e^{2X_n}, which can be rewritten as a hyperbolic tangent function of the headway $\Delta x_n(t)$. Performing this elimination, we obtain a differential-difference equation that the form (B2) satisfies:

$$\dot{x}_n(t+\tau)=C+A\nu\frac{N_+ + N_- e^{2X_n}}{D_+ + D_- e^{2X_n}}, \quad \text{(B10)}$$

where

$$D_\pm=\vartheta_0^2(0)\vartheta_1(\beta)\vartheta_1(2\delta\pm\beta),$$
$$N_\pm=\pm\{\vartheta_1^2(\delta\pm\beta)[\vartheta_0^2(\delta)]'-\vartheta_0^2(\delta\pm\beta)[\vartheta_1^2(\delta)]'\}. \quad \text{(B11)}$$

We can transform $N_\pm$ here into expressions involving differentiation with respect to β rather than δ:

$$N_\pm=\left(\pm\frac{d}{d\delta}-\frac{d}{d\beta}\right)[\vartheta_1^2(\delta\pm\beta)\vartheta_0^2(\delta)-\vartheta_0^2(\delta\pm\beta)\vartheta_1^2(\delta)]$$
$$=\pm\vartheta_0^2(0)\vartheta_1(\beta)\vartheta_1(2\delta\pm\beta)\frac{d}{d\beta}\ln\frac{\vartheta_1(2\delta\pm\beta)}{\vartheta_1(\beta)}. \quad \text{(B12)}$$

As mentioned above, Eq. (B10) can be written by using the hyperbolic tangent function as

$$\dot{x}_n(t+\tau)=C+\frac{A\nu}{2}\left(\frac{N_-}{D_-}+\frac{N_+}{D_+}\right)+\frac{A\nu}{2}\left(\frac{N_-}{D_-}-\frac{N_+}{D_+}\right)\times\tanh\left(X_n+\frac{1}{2}\ln\frac{D_-}{D_+}\right). \quad \text{(B13)}$$

Thus, we find the equations to which the parameters are subject by comparing the above equation with the equation of motion (B1). We obtain

$$\xi=C+\frac{A\nu}{2}\frac{d}{d\beta}\ln\frac{\vartheta_1(2\delta+\beta)}{\vartheta_1(2\delta-\beta)}, \quad \text{(B14)}$$

$$\eta=\frac{A\nu}{2}\frac{d}{d\beta}\ln\frac{\vartheta_1^2(\beta)}{\vartheta_1(2\delta+\beta)\vartheta_1(2\delta-\beta)}, \quad \text{(B15)}$$

$$\rho=h-A\ln\frac{\vartheta_1(2\delta-\beta)}{\vartheta_1(2\delta+\beta)}, \quad \text{(B16)}$$

$$\sigma=A. \quad \text{(B17)}$$

In the first two equations, we can decompose the variables δ and β in the theta functions using the addition formulas. We obtain the following:

$$\xi=C+\frac{A\nu}{2}\frac{d}{d\delta}\ln\vartheta_0(2\delta)+\frac{A\nu}{4}\frac{d}{d\delta}\ln\left(\frac{\vartheta_1^2(2\delta)}{\vartheta_0^2(2\delta)}-\frac{\vartheta_1^2(\beta)}{\vartheta_0^2(\beta)}\right), \quad \text{(B18)}$$

$$\eta=-\frac{A\nu}{2}\frac{d}{d\beta}\ln\left(\frac{\vartheta_0^2(\beta)}{\vartheta_1^2(\beta)}-\frac{\vartheta_0^2(2\delta)}{\vartheta_1^2(2\delta)}\right). \quad \text{(B19)}$$

For the first of these, $d/d\beta$ is converted into $d/d\delta$, to apply the addition formulas, before the decomposition. It can be shown that the relations (B14)–(B17) are equivalent to those that we found previously [9], although they are quite different in appearance.

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